Taylor and Maclaurin Series

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems: Which functions have power series representations? How can we find such representations?

We start by supposing that f is any function that can be represented by a power series:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots \qquad |x-a| < R \tag{1}$$

Let's try to determine what the coefficients c_n must be in terms of f. To begin, notice that if we put $x = a$ in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

By Theorem 8.6.2, we can differentiate the series in Equation 1 term by term:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots \qquad |x-a| < R \tag{2}$$

and substitution of $x = a$ in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \dots \qquad |x-a| < R \tag{3}$$

Again we put $x = a$ in Equation 3. The result is

$$f''(a) = 2c_2$$

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \dots \qquad |x-a| < R \tag{4}$$

and substitution of $x = a$ in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \dots \cdot nc_n = n!c_n$$

Solving this equation for the nth coefficient c_n, we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$.

Thus we have proved the following theorem.

THEOREM: If f has a power series representation (expansion) at a, that is, if

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{and} \quad |x-a|<R, \quad \text{then} \quad c_n=\frac{f^{(n)}(a)}{n!} \tag{5}$$

Substituting this formula for c_n back into the series, we see that *if* f has a power series expansion at a, then it must be of the following form.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots \end{aligned} \tag{6}$$

The series in Equation 6 is called the **Taylor series of the function f at a (or about a or centered at a).** For the special case $a=0$ the Taylor series becomes

$$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots \tag{7}$$

This case arises frequently enough that is is given the special name **Maclaurin series**.

NOTE: We have shown that *if* f can be represented as a power series about a, then f is equal to the sum of its Taylor series. But there exist functions that are not equal to the sum of their Taylor series. For example, one can show that the function defined by

$$f(x)=\begin{cases} e^{-1/x^2} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

is not equal to its Maclaurin series.

EXAMPLE 1: Find the Maclaurin series of the function $f(x)=e^x$ and its radius of convergence.

Solution: If $f(x)=e^x$, then $f^{(n)}(x)=e^x$, so $f^{(n)}(0)=e^0=1$ for all n. Therefore, the Taylor series for f at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

To find the radius of convergence we let $a_n=x^n/n!$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0<1$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is $R=\infty$.

The conclusion we can draw from (5) and Example 1 is that *if* e^x has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So how can we determine whether e^x *does* have a power series representation?

Let's investigate the more general question: Under what circumstances is a function equal to the sum of its Taylor series? In other words, if f has derivatives of all orders, when is it true that

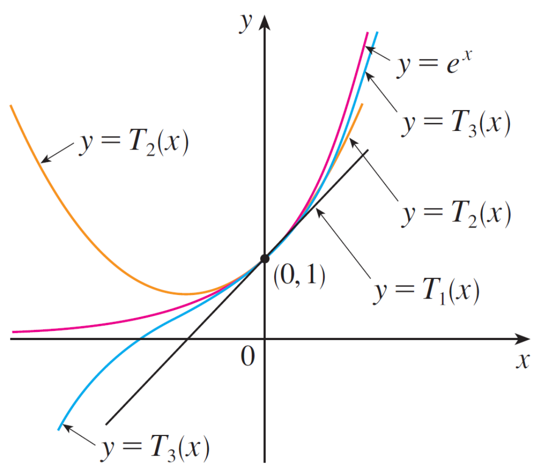
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$T_n(x) \quad = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that T_n is a polynomial of degree n called the **nth-degree Taylor polynomial of f at a.** For instance, for the exponential function $f(x) = e^x$, the result of Example 1 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1, 2$, and 3 are

$$T_1(x) = 1 + x, \qquad T_2(x) = 1 + x + \frac{x^2}{2!}, \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$



In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \quad \text{so that} \quad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

We have therefore proved the following.

THEOREM: If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0 \tag{8}$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

In trying to show that $\lim\limits_{n\to\infty} R_n(x) = 0$ for a specific function f, we usually use the expression in the next theorem.

THEOREM (TAYLOR'S FORMULA): If f has $n+1$ derivatives in an interval I that contains the number a, then for x in I there is a number z strictly between x and a such that the remainder term in the Taylor series can be expressed as

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

NOTE 1: For the special case $n = 0$, if we put $x = b$ and $z = c$ in Taylor's Formula, we get

$$f(b) = f(a) + f'(c)(b-a) \tag{9}$$

which is the Mean Value Theorem. In fact, Theorem 9 can be proved by a method similar to the proof of the Mean Value Theorem.

NOTE 2: Notice that the remainder term

$$\boxed{R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}} \tag{10}$$

is very similar to the terms in the Taylor series except that $f^{(n+1)}$ is evaluated at z instead of at a. All we say about the number z is that it lies somewhere between x and a. The expression for $R_n(x)$ in Equation 10 is known as **Lagrange's form of the remainder term.**

NOTE 3: In Section 8.8 we will explore the use of Taylor's Formula in approximating functions. Our immediate use of it is in conjunction with Theorem 8.

In applying Theorems 8 and 9 it is often helpful to make use of the following fact:

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{for every real number } x} \tag{11}$$

This is true because we know from Example 1 that the series $\sum \frac{x^n}{n!}$ converges for all x and so its nth term approaches 0.

EXAMPLE 2: Prove that e^x is equal to the sum of its Taylor series with $a = 0$ (Maclaurin series).

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Solution: If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$, so the remainder term in Taylor's Formula is

$$R_n(x) = \frac{e^z}{(n+1)!}x^{n+1}$$

where z lies between 0 and x. (Note, however, that z depends on n.) If $x > 0$, then $0 < z < x$, so $e^z < e^x$. Therefore

$$0 < R_n(x) = \frac{e^z}{(n+1)!}x^{n+1} < e^x \frac{x^{n+1}}{(n+1)!} \to 0$$

by Equation 11, so $R_n(x) \to 0$ as $n \to \infty$ by the Squeeze Theorem. If $x < 0$, then $x < z < 0$, so $e^z < e^0 = 1$ and

$$|R_n(x)| < \frac{|x|^{n+1}}{(n+1)!} \to 0$$

Again $R_n(x) \to 0$. Thus, by (8), e^x is equal to the sum of its Maclaurin series, that is,

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x} \tag{12}$$

In particular, if we put $x = 1$ in Equation 12, we obtain

$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots} \tag{13}$$

EXAMPLE 3: Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Solution: We have $f^{(n)}(2) = e^2$ and so, putting $a = 2$ in the definition of a Taylor series (6), we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

Again it can be verified, as in Example 1, that the radius of convergence is $R = \infty$. As in Example 2 we can verify that $\lim\limits_{n\to\infty} R_n(x) = 0$, so

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \quad \text{for all } x \tag{14}$$

We have two power series expansions for e^x, the Maclaurin series in Equation 12 and the Taylor series in Equation 14. The first is better if we are interested in values of x near 0 and the second is better if x is near 2.

EXAMPLE 4: Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

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Solution: We arrange our computation in two columns as follows:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\begin{aligned} f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots &= 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \ldots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

Using the remainder term (10) with $a = 0$, we have

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$$

where $f(x) = \sin x$ and z lies between 0 and x. But $f^{(n+1)}(z)$ is $\pm \sin z$ or $\pm \cos z$. In any case, $|f^{(n+1)}(z)| \leq 1$ and so

$$0 \leq |R_n(x)| = \frac{|f^{(n+1)}(z)|}{(n+1)!}|x^{n+1}| \leq \frac{1}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!} \tag{15}$$

By Equation 11 the right side of this inequality approaches 0 as $n \to \infty$, so $R_n(x) \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so $\sin x$ is equal to the sum of its Maclaurin series by Theorem 8. Thus

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x} \tag{16}$$

EXAMPLE 5: Find the Maclaurin series for $\cos x$.

EXAMPLE 5: Find the Maclaurin series for $\cos x$.

Solution 1: We arrange our computation in two columns as follows:

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \end{aligned}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\begin{aligned} f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots &= 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \ldots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \end{aligned}$$

Solution 2: We differentiate the Maclaurin series for $\sin x$ given by Equation 16:

$$\begin{aligned} \cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \\ &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \ldots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \end{aligned}$$

Since the Maclaurin series for $\sin x$ converges for all x, Theorem 8.6.2 tells us that the differentiated series for $\cos x$ also converges for all x. Thus

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad \text{for all } x} \tag{17}$$

EXAMPLE 6: Find the Maclaurin series for $f(x) = x\cos x$.

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Solution: Instead of computing derivatives and substituting in Equation 7, it's easier to multiply the series for $\cos x$ (Equation 17) by x:

$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!}$$

REMARK: The power series that we obtained by indirect methods in Examples 5 and 6 and in Section 8.6 are indeed the Taylor or Maclaurin series of the given functions because Theorem 5 asserts that, no matter how we obtain a power series representation $f(x) = \sum c_n(x-a)^n$, it is always true that $c_n = f^{(n)}(a)/n!$. In other words, the coefficients are uniquely determined.

EXAMPLE 7: Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Solution: Arranging our work in columns, we have

$$\begin{array}{ll}
f(x) = (1+x)^k & f(0) = 1 \\
f'(x) = k(1+x)^{k-1} & f'(0) = k \\
f''(x) = k(k-1)(1+x)^{k-2} & f''(0) = k(k-1) \\
f'''(x) = k(k-1)(k-2)(1+x)^{k-3} & f'''(0) = k(k-1)(k-2) \\
\vdots & \vdots \\
f^{(n)}(x) = k(k-1)\dots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) = k(k-1)\dots(k-n+1)
\end{array}$$

Therefore, the Maclaurin series of $f(x) = (1+x)^k$ is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = 1 + \sum_{n=1}^{\infty}\frac{k(k-1)\dots(k-n+1)}{n!}x^n$$

This series is called the **binomial series**. If its nth term is a_n, then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{k(k-1)\dots(k-n+1)(k-n)x^{n+1}}{(n+1)!}\cdot\frac{n!}{k(k-1)\dots(k-n+1)x^n}\right| \\
&= \frac{|k-n|}{n+1}|x| = \frac{\left|\frac{k}{n}-1\right|}{1+\frac{1}{n}}|x| \to |x| \quad \text{as } n\to\infty
\end{aligned}$$

Thus by the Ratio Test the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{(k-n)!(k-n+1)\dots(k-2)(k-1)k}{n!(k-n)!} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}$$

and these numbers are called the **binomial coefficients.**

The following theorem states that $(1+x)^k$ is equal to the sum of its Maclaurin series. It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0, but that turns out to be quite difficult.

THEOREM (THE BINOMIAL SERIES): If k is any real number and $|x|<1$, then

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots} \qquad (18)$$

Although the binomial series always converges when $|x|<1$, the question of whether or not it converges at the endpoints, ± 1, depends on the value of k. It turns out that the series converges at 1 if $-1<k<0$ and at both endpoints if $k \geq 0$. Notice that if k is a positive integer and $n>k$, then the expression for $\binom{k}{n}$ contains a factor $(k-k)$, so

$$\binom{k}{n} = 0$$

for $n>k$. This means that the series terminates and reduces to the ordinary Binomial Theorem when k is a positive integer.

EXAMPLE 8: Find the Maclaurin series for $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.

Solution: We write $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Using the binomial series with $k = -\dfrac{1}{2}$ and with x replaced by $-\dfrac{x}{4}$, we have

$$\begin{aligned}
\frac{1}{\sqrt{4-x}} &= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n \\
&= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{x}{4}\right)^3\right. \\
&\qquad \left. + \ldots + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\ldots(-\frac{1}{2}-n+1)}{n!}\left(-\frac{x}{4}\right)^n + \ldots\right] \\
&= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1\cdot 3}{2!8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!8^3}x^3 + \ldots + \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{n!8^n}x^n + \ldots\right]
\end{aligned}$$

We know from (18) that this series converges when $|-x/4|<1$, that is, $|x|<4$, so the radius of convergence is $R=4$.

We collect in the following table, for future reference, some important Maclaurin series that we have derived in this section and the preceding one.

$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$	$R = 1$
$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$R = \infty$
$\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$R = \infty$
$\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$R = \infty$
$\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$R = 1$
$\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$R = 1$
$(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$	$R = 1$

One reason that Taylor series are important is that they enable us to integrate functions that we couldn't previously handle. In fact, the function $f(x) = e^{-x^2}$ can't be integrated by techniques discussed so far because its antiderivative is not an elementary function (see Section 6.4). In the following example we write f as the Maclaurin series to integrate this function.

EXAMPLE 9:

(a) Evaluate $\displaystyle\int e^{-x^2}\,dx$ as an infinite series.

(b) Evaluate $\displaystyle\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

EXAMPLE 9:

(a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Solution:

(a) First we find the Maclaurin series for $f(x) = e^{-x^2}$. Although it's possible to use the direct method, let's find it simply by replacing x with $-x^2$ in the series for e^x given in the table of Maclaurin series. Thus, for all values of x,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

Now we integrate term by term:

$$\begin{aligned}\int e^{-x^2}\,dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + (-1)^n \frac{x^{2n}}{n!} + \dots\right) dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \dots\end{aligned}$$

This series converges for all x because the original series for e^{-x^2} converges for all x.

(b) The Fundamental Theorem of Calculus gives

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \dots\right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \dots \\ &\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475\end{aligned}$$

The Alternating Series Estimation Theorem shows that the error involved in this approximation is less than

$$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001$$

Another use of Taylor series is illustrated in the next example. The limit could be found with l'Hospital's Rule, but instead we use a series.

EXAMPLE 10: Evaluate $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

EXAMPLE 10: Evaluate $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}$.

Solution: Using the Maclaurin series for e^x, we have

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1-x}{x^2} &= \lim_{x\to 0}\frac{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)-1-x}{x^2}\\ &= \lim_{x\to 0}\frac{\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots}{x^2}\\ &= \lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\frac{x^3}{5!}+\dots\right)=\frac{1}{2}\end{aligned}$$

because power series are continuous functions.