Bernoulli's inequality **(for integer cases)**

$$(1 + x)^n \geq 1 + nx \qquad (1)$$

with conditions:

(a) (1) is true $\forall\ n\in \mathbf{N} \cup \{0\}$ and $\forall x\in \mathbf{R}$, $x \geq -1$.

(b) (1) is true $\forall\ n\in \mathbf{2N}$ and $\forall x\in \mathbf{R}$.

The strict inequality: $(1 + x)^n > 1 + nx \qquad (2)$

is true for every integer $n \geq 2$ and every real number $x \geq -1$ with $x \neq 0$.

(The strict inequality is not discussed in the following.)

Proof 1 Use Mathematical Induction

Condition **(a)**

Let P(n) be the proposition: $(1 + x)^n \geq 1 + nx$ $\quad \forall n\in \mathbf{N} \cup \{0\}$ and $\forall x\in \mathbf{R}$, $x \geq -1$.

For P(0), $(1 + x)^0 = 1 \geq 1 + 0x$ $\quad \therefore$ P(0) is true.

Assume P(k) is true for some $k\in \mathbf{Z}$, $k \geq 0$,

that is, $(1 + x)^k \geq 1 + kx$ $\quad \forall\ k\in \mathbf{N} \cup \{0\}$ and $\forall x\in \mathbf{R}$, $x \geq -1$. (3)

For P(k+1), $(1 + x)^{k+1} = (1 + x)^k(1 + x)$

$\geq (1 + kx)(1 + x)$, by (3) and also note that since $x \geq -1$, the factor $(x + 1) \geq 0$.

$= 1 + (k + 1)x + kx^2$

$\geq 1 + (k + 1)x$

$\therefore$ P(k+1) is also true.

By the Principle of Mathematical Induction, P(n) is true $\forall n\in \mathbf{N} \cup \{0\}$ and $\forall x\in \mathbf{R}$, $x \geq -1$.

Condition **(b)**

Let P(n) be the proposition: $(1 + x)^n \geq 1 + nx$ $\quad \forall\ n\in \mathbf{2N}$ and $\forall x\in \mathbf{R}$.

For P(0), $(1 + x)^0 = 1 \geq 1 + 0x$ $\quad \therefore$ P(0) is true.

For P(2), $(1 + x)^2 = 1 + 2x + x^2 \geq 1 + 2x$, since $x^2 \geq 0$, $\forall x\in \mathbf{R}$.

$\therefore$ P(2) is true.

Assume P(k) is true for some $k\in \mathbf{Z}$, $k \geq 0$,

that is, $(1 + x)^k \geq 1 + kx$ $\quad \forall\ k\in \mathbf{N} \cup \{0\}$ and $\forall x\in \mathbf{R}$, $x \geq -1$. (4)

For P(k + 2),

$(1 + x)^{k+2} = (1 + x)^k (1 + x)^2 \geq (1 + kx)(1 + 2x)$, by (4) and P(2).

$= 1 + (k + 2)x + 2kx^2$

$\geq 1 + (k + 2)x$, since $k > 0$ and $x^2 \geq 0$, $\forall x\in \mathbf{R}$.

$\therefore$ P(k + 2) is also true.

By the Principle of Mathematical Induction, P(n) is true $\forall\ n\in \mathbf{2N}$ and $\forall x\in \mathbf{R}$.

Condition **(a)** is discussed only in the following.

Proof 2 Use A.M. ≥ G.M.

Consider the A.M. and G.M. of n positive numbers (1 + nx), 1, 1, ….,1 [with (n-1) “1”s]

$$\frac{(1+nx)+1+1+....+1}{n} \geq \sqrt[n]{(1+nx).1.1....1}$$

$$\frac{n+nx}{n} \geq \sqrt[n]{(1+nx)}$$

$$\therefore \quad (1+x)^n \geq 1+nx \qquad (1)$$

Note : Numbers should be positive before applying A.M. – G.M. theorem.

In the numbers used in A.M.-G.M. above, 1 > 0 and $1 + nx \geq 0$, i.e. $x \geq -\frac{1}{n}$.

However, if $1 + nx < 0$, since it is given that $x \geq -1$, or $x + 1 \geq 0$,

L.H.S. of (1) = $(1 + x)^n \geq 0$

R.H.S. of (1) = $1 + nx < 0$

$\therefore$ $(1 + x)^n \geq 0 > 1 + nx$, which is always true.

$\therefore$ $x \geq -1$ and not $x \geq -\frac{1}{n}$ can ensure (1) is correct.

Proof 3 Use Binomial Theorem

(a) For x > 0, $(1+x)^n = 1+nx+\binom{n}{2}x^2+\binom{n}{3}x^3+\binom{n}{4}x^4+....+\binom{n}{n}x^n \geq 1+nx$ (5)

(b) For x = 0, obviously $(1 + x)^n \geq 1 + nx$ is true.

(c) For $-1 < x < 0$, (The proof below is not very rigorous.)

Put $y = -x$, then $0 < y < 1$,

$$(1-y)^n = 1-ny+\binom{n}{2}y^2-\binom{n}{3}y^3+\binom{n}{4}y^4-....+(-1)^n\binom{n}{n}y^n \qquad (6)$$

Put y = 0, we have:

$$0=(1-1)^n = 1-n+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-....+(-1)^n\binom{n}{n}$$

$$\therefore \binom{n}{2}-\binom{n}{3}+\binom{n}{4}-....+(-1)^n\binom{n}{n} = n-1>0 \qquad (7)$$

Now $0 < y < 1$, $\therefore$ $y^2 > y^3 > \ldots > y^n$

Therefore in (7), each term is multiplied by a factor that is smaller than the term before,

$$\binom{n}{2}y^2-\binom{n}{3}y^3+\binom{n}{4}y^4-....+(-1)^n\binom{n}{n}y^n \geq 0 \qquad (8)$$

From (6),

$(1 - y)^n \geq 1 - ny$ for $0 < y < 1$.

or $(1 + x)^n \geq 1 + nx$ for $-1 < x < 0$.

Extension of Bernoulli's inequality

Given $x > -1$, then

(a) $(1 + x)^r \le 1 + rx$ for $0 < r < 1$ (9)

(b) $(1 + x)^r \ge 1 + rx$ for $r < 0$ or $r > 1$ (10)

Firstly we give the proof that r is **a rational number** first.

Proof 4 Use A.M. ≥ G.M.

Since $r \in \mathbf{Q}$, $r = \dfrac{p}{q}$

(a) Let $0 < r < 1$, $\therefore$ $p < q$, $q - p > 0$. Also $1 + x > 0$,

$$(1 + x)^r = (1+x)^{p/q} = \sqrt[q]{(1+x)^p 1^{q-p}} \le \frac{p(1+x) + 1\cdot(q-p)}{q} \qquad \text{(G.M.} \le \text{A.M.)}$$

$$= \frac{px+q}{q} = 1 + \frac{p}{q}x = 1 + rx$$

(b) Let $r > 1$,

(i) If $1 + rx \le 0$, then $(1 + x)^r > 0 \ge 1 + rx$.

(ii) If $1 + rx > 0$, $rx > -1$.

Since $r > 1$, we have $0 < \dfrac{1}{r} < 1$. By (a) we get: $(1+rx)^{1/r} \le 1 + \dfrac{1}{r}rx = 1 + x$

$\therefore$ $(1 + x)^r \ge 1 + rx$.

Let $r < 0$, then $-r > 0$.

Choose a natural number n sufficiently large such that $0 < -r/n < 1$ and $1 > rx/n > -1$.

By (a), $0 < (1+x)^{-r/n} \le 1 + \dfrac{r}{n}x > 0$.

$$\text{Since} \quad 1 \ge 1 - \left(\frac{r}{n}x\right)^2 = \left(1 - \frac{r}{n}x\right)\left(1 + \frac{r}{n}x\right) \Rightarrow \left(1 - \frac{r}{n}x\right)^{-1} \ge 1 + \frac{r}{n}x \qquad (11)$$

Hence by (11),

$$(1+x)^r \ge \left(1 + \frac{r}{n}x\right)^n \ge 1 + n\left(\frac{r}{n}x\right) = 1 + rx.$$

Again, equality holds if and only if $x = 0$.

Note : If r is **irrational**, we choose an infinite sequence of rational numbers r_1, r_2, r_3, …., such that r_n tends to r as n tends to infinity. For part (a), we can extend to irrational r:

$$(1+x)^r = \lim_{n\to\infty}(1+x)^{r_n} \le \lim_{n\to\infty}(1 + r_n x) = 1 + rx.$$

Similar argument for part (b) completes the proof for the case where $r \in \mathbf{R}$.

Proof 5 Use analysis

Let $f(x) = (1 + x)^r - 1 - rx$ where $x > -1$ and $r \in \mathbf{R} \setminus \{0, 1\}$ (12)

Then f(x) is differentiable and its derivative is:

$$f'(x) = r(1 + x)^{r-1} - r = r\,[(1 + x)^{r-1} - 1] \quad (13)$$

from (13) we can get $f'(x) = 0 \Leftrightarrow x = 0$.

(a) If $0 < r < 1$, then $f'(x) > 0 \;\; \forall\, x \in (-1, 0)$ and $f'(x) < 0 \;\; \forall\, x \in (0, +\infty)$.

$\therefore$ $x = 0$ is a ***global maximum*** point of f.

$\therefore$ $f(x) < f(0) = 0$.

$\therefore$ $(1 + x)^r \leq 1 + rx$ for $0 < r < 1$.

(b) If $r < 0$ or $r > 1$, then $f'(x) < 0 \;\; \forall\, x \in (-1, 0)$ and $f'(x) > 0 \;\; \forall\, x \in (0, +\infty)$.

$\therefore$ $x = 0$ is a ***global minimum*** point of f.

$\therefore$ $f(x) > f(0) = 0$.

$\therefore$ $(1 + x)^r \geq 1 + rx$ for $r < 0$ or $r > 1$.

Finally, please check that the equality holds for $x = 0$ or for $r \in \{0, 1\}$. The proof is complete.